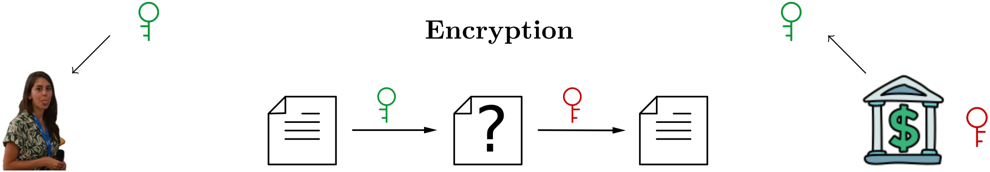
Open problems in code-based cryptography

Violetta Weger

Coding Theory Colloquium

November 8, 2023

Gentle Introduction to Crypto



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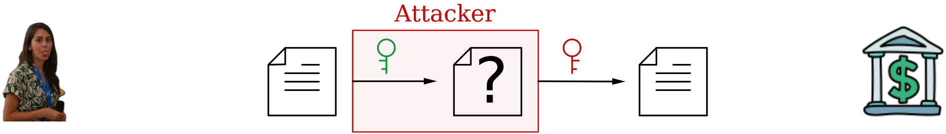
Encryption



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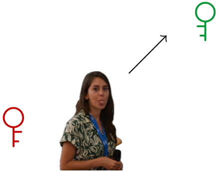

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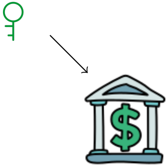

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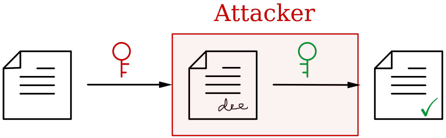


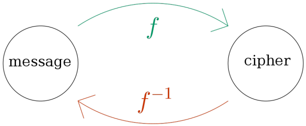
Gentle Introduction to Crypto

Encryption

Trapdoor

- f easy to compute with
- f^{-1} hard to compute with

→ f^{-1} easy with secret



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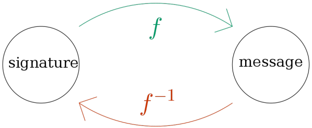

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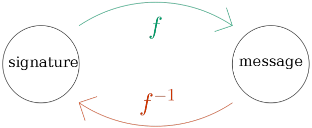

Attacks

- message recovery/forgery: f^{-1} without secret
- key recovery: find secret

→ security level λ: best algo. needs 2^{λ} ops.

Constraints

- public, signature **small**
- decrypt/sign:

  f^{-1} with secret **fast**

Classic Crypto vs. Post-quantum Crypto

Classic crypto

- Integer factorization: $f(p, q) = p \cdot q = n$
- DLP: $f(c) = a^c = b$ in $\mathbb{F}_q$ or ell. curve

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- ∘ quantum algos. need exp. time

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- ∘ 2016: 3 code-based encryption

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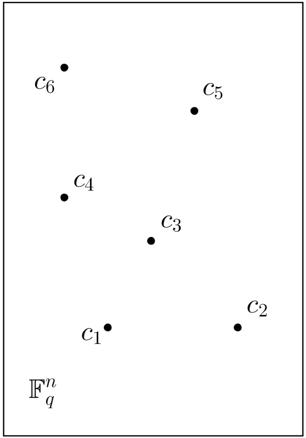
Today's talk: open questions for these schemes

- ∘ Classic McEliece
- ∘ McEliece signature
- ∘ decoding rank-metric codes
- ∘ code equivalence problems

Coding Theory



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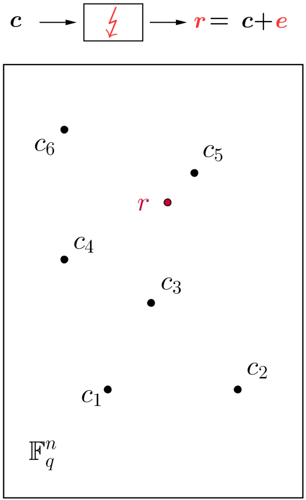

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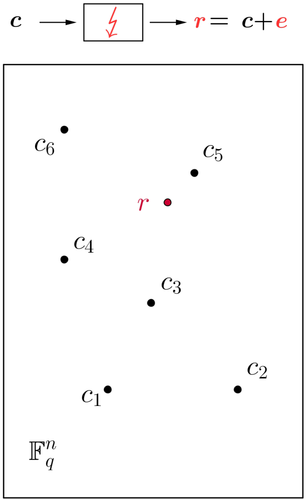

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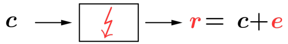

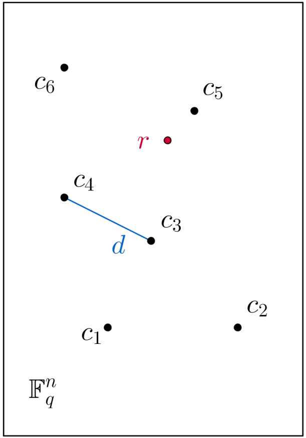

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$$d(\mathcal{C}) = \min\{d_H(x, y) \mid x \neq y \in \mathcal{C}\}$$

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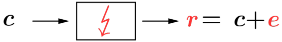

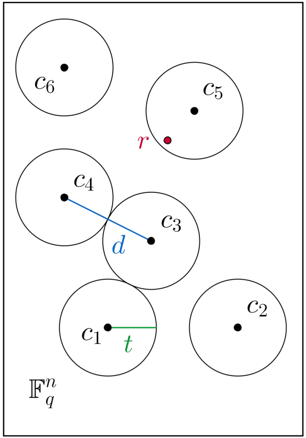

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- *error-correction capability*: $t = \lfloor (d(\mathcal{C}) - 1)/2 \rfloor$

Syndrome Decoding Problem

Syndrome Decoding Problem (SDP):
Given p.c. matrix $H \in \mathbb{F}_q^{(n-k)\times n}$, syndrome $s \in \mathbb{F}_q^{n-k}$, target weight t, find $e \in \mathbb{F}_q^n$ s.t.

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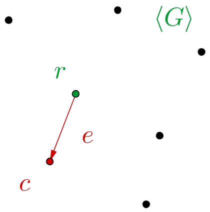

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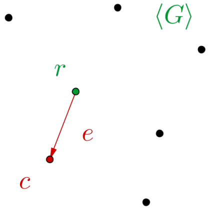

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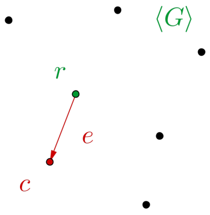

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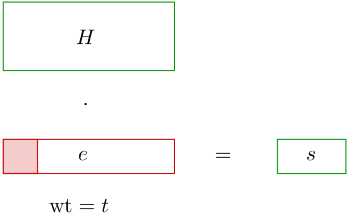

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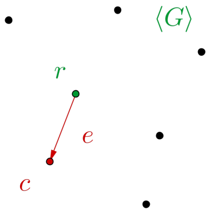

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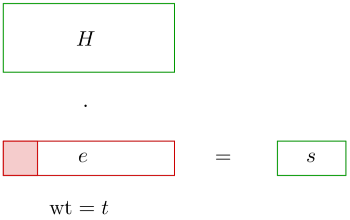

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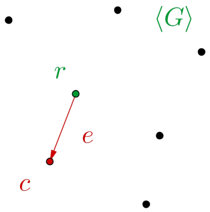

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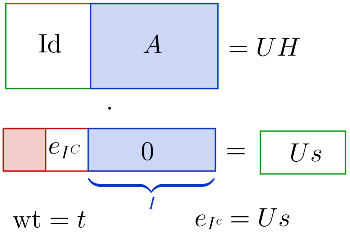
- find error-free information set I
- cost $= \binom{n}{t}\binom{n-k}{t}^{-1}$
- assume $t = (d-1)/2$, d from GV

→ cost $q^{nf(n,R)} \sim 2^{0.05n}$



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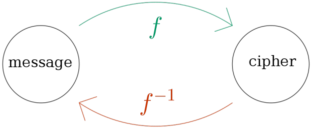

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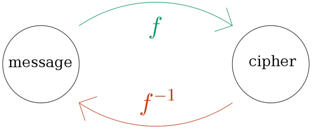

Encryption scheme

- secret H (with efficient decoder)
- public scrambled H', s, t
- → cipher = f(message)

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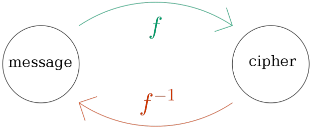

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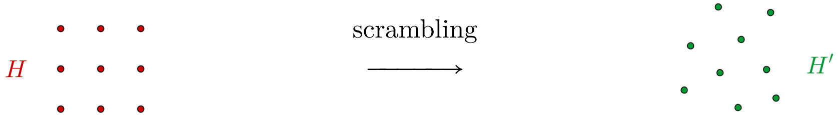
Which secret code to choose?

Classic McEliece

- secret H Goppa code
- public $H' = SHP$, S invertible, P permutation



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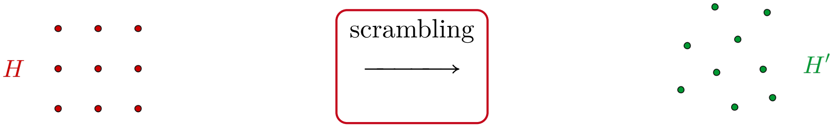

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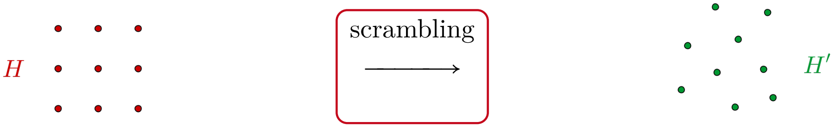

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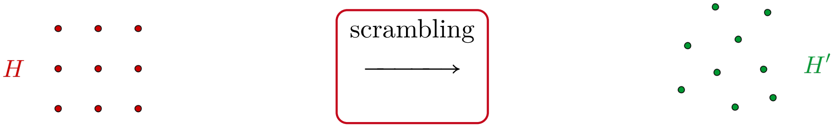

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TII McEliece Challenges:

Theoretical key recovery: 10'000 $

Distinguish Goppa Codes

Famous distinguisher: Square code

- $a, b \in \mathbb{F}_q^n : a \star b = (a_1 b_1, \ldots, a_n b_n)$
- $\mathcal{C}^{(2)} = \langle \{a \star b : a, b \in \mathcal{C}\} \rangle$

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$\mathsf{GRS}(\alpha, \beta) = \{(f(\alpha_1)\beta_1, \dots, f(\alpha_n)\beta_n) : f \in \mathbb{F}_{p^m}[x], \deg < k\}$

$$H = \begin{pmatrix} \beta_1' & \beta_2' & \cdots & \beta_n' \\ \alpha_1\beta_1' & \alpha_2\beta_2' & \cdots & \alpha_n\beta_n' \\ \vdots & \vdots & & \vdots \\ \alpha_1^{t-1}\beta_1' & \alpha_2^{t-1}\beta_2' & \cdots & \alpha_n^{t-1}\beta_n' \end{pmatrix} \in \mathbb{F}_{p^m}^{t \times n}$$

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$\mathsf{GRS}(\alpha, \beta) \in \mathbb{F}_{p^m}^n$ of dim $k = n - t$ $\to$ $\dim(\mathsf{GRS}(\alpha, \beta)^{(2)}) = \min\{2k - 1, n\}$

Distinguish Goppa Codes

$\mathsf{Goppa}(\alpha, g) \in \mathbb{F}_p^n$ of dim $k' = n - tm$

- $g \in \mathbb{F}_{p^m}[x]$ irreducible
- $\beta_i = g(\alpha_i)^{-1}$

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Distinguish Goppa Codes

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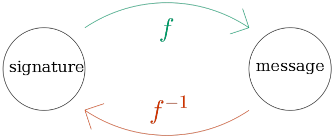
Goal: Given $SH_{\mathsf{Goppa}(\alpha,g)}P$ recover α, g

Hash & Sign

SDP:
Given H, s, t find e with
1. $s = eH^\top$ 2. $\mathrm{wt}_H(e) \le t$



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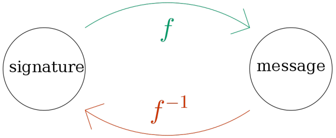

Signature scheme

- secret H (with efficient decoder)
- public scrambled $H' = HP, s, t$
- → signature $\sigma = f^{-1}$(message)

$$f : \{e \in \mathbb{F}_q^n : \mathrm{wt}(e) \leq t\} \to \mathbb{F}_q^{n-k},$$
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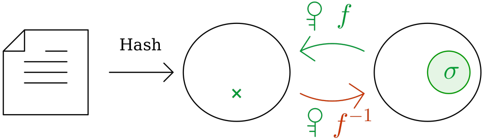

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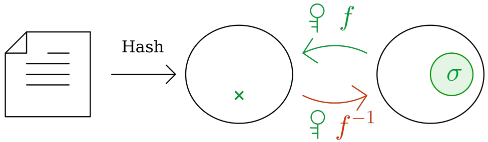

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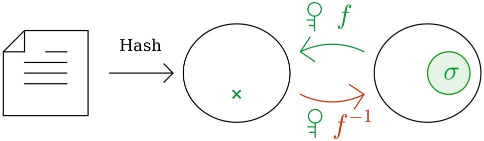

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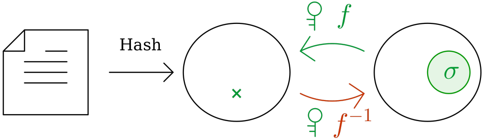

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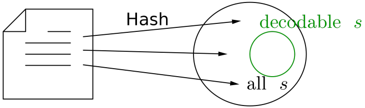

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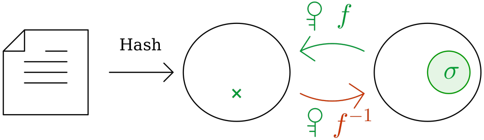

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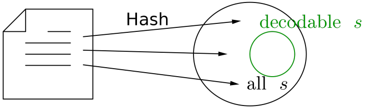
- → salting the Hash → slow



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$\mathsf{Hash}(m)$ not decodable syndrome → Hash & Sign scheme: slow

Decodable syndromes $\sum_{i=0}^{t} \binom{n}{i}(q-1)^i$ $<$ q^{n-k} All syndromes

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Need an almost perfect code!

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2. Open problem: Find code

- efficiently decodable
- almost perfect
- not distinguishable

Rank Metric

$\mathbb{F}_{q^m}$-linear codes

- $\mathcal{C} \subset \mathbb{F}_{q^m}^n$
- $\mathcal{C} = \langle G \rangle$, $G \in \mathbb{F}_{q^m}^{k \times n}$
- codewords $c = xG$ for $x \in \mathbb{F}_{q^m}^k$

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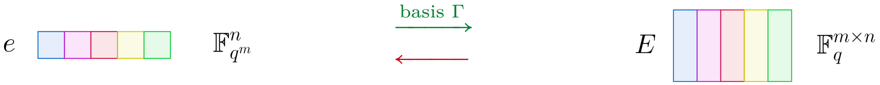

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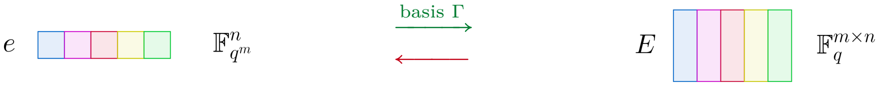

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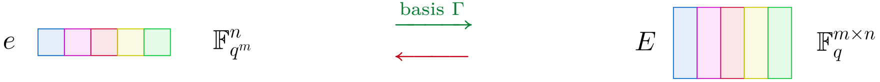

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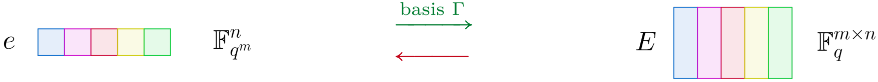

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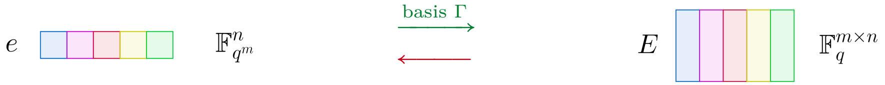

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- Used in: ROLLO, RQC, RYDE

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- Used in: MIRA, MiRitH

Hardness of Rank SDP

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How to show a problem $\mathcal{P}$ is NP-hard?

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Reduction from $\mathcal{Q}$ to $\mathcal{P}$:

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Reduction from $\mathcal{Q}$ to $\mathcal{P}$:

→ If we can solve $\mathcal{P}$
  → we can solve $\mathcal{Q}$

→ hardness $\mathcal{P} \geq$ hardness $\mathcal{Q}$

→ $\mathcal{P}$ is NP-hard

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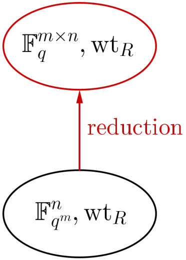
Connections to Other Problems



1. expand code $\mathcal{C}$ to $\mathbb{F}_q^{m\times n}$ via basis Γ → $d_R(\Gamma(\mathcal{C})) = d_R(\mathcal{C})$

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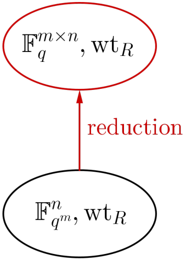

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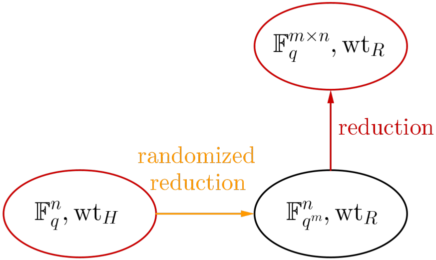

$\alpha \in \mathbb{F}_{q^m}^n$ lin. indep.

$\psi_\alpha : x \mapsto \alpha \star x$

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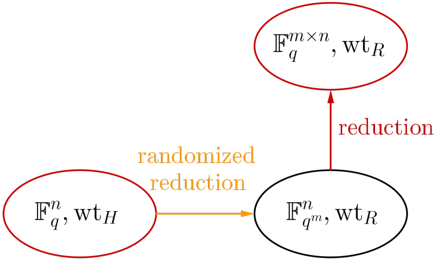
Connections to Other Problems



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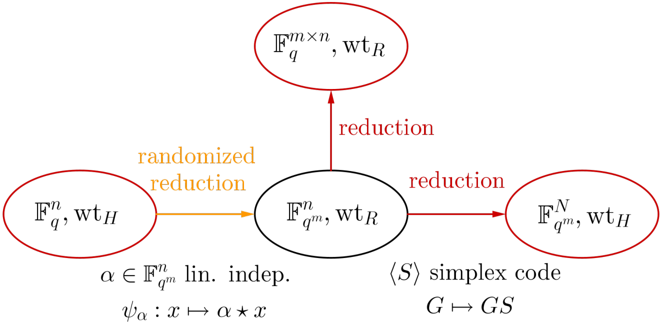

3. $S \in \mathbb{F}_q^{n\times N}$, $N = \frac{q^n-1}{q-1}$ $\quad\rightarrow\quad d_R(\langle G\rangle) = d_H(\langle GS\rangle)$

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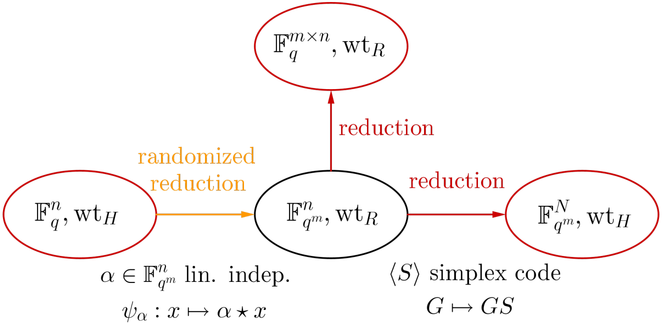
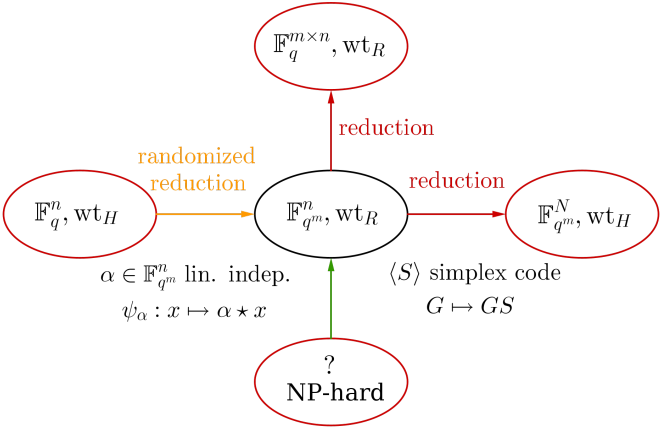


3. $S \in \mathbb{F}_q^{n\times N}$, $N = \frac{q^n-1}{q-1}$ → $d_R(\langle G\rangle) = d_H(\langle GS\rangle)$ → hardness $(\mathbb{F}_{q^m}^N, \mathrm{wt}_H) \geq (\mathbb{F}_{q^m}^n, \mathrm{wt}_R)$

Connections to Other Problems



→ new NP-hard problem to reduce from?

How to solve Rank SDP

More costly than Hamming ISD? $\begin{bmatrix} n \\ t \end{bmatrix}_q > \binom{n}{t}$

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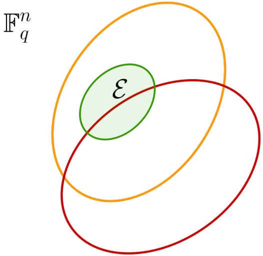

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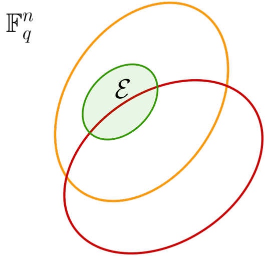

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Combinatorial solver:

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- if $t \sim \log(n)$, $T = t/\log(n)$
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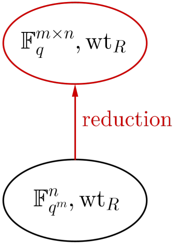

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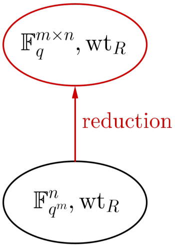

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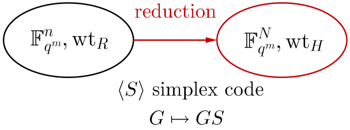
Hamming ISD:

→ $q^{n^2 RT}$ optimal



Other solvers?

Code Equivalence

linear isometry: $\mathrm{wt}(x) = \mathrm{wt}(\varphi(x)) \ \forall x \to$ Hamming metric: $(\mathbb{F}_q^\star)^n \rtimes S_n \times \mathrm{Aut}(\mathbb{F}_q)$

code equivalence: $\mathcal{C} \sim \mathcal{C}'$ if exists lin. isometry φ: $\varphi(\mathcal{C}) = \mathcal{C}'$

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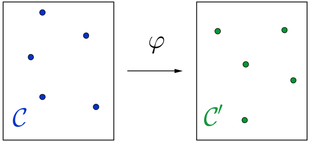
Linear Equivalence Problem (LEP)

Given $G, G' \in \mathbb{F}_q^{k \times n}$, find $\varphi \in (\mathbb{F}_q^\star)^n \rtimes S_n$
s.t. $\varphi(\langle G \rangle) = \langle G' \rangle$.



LEP

- LEP used in: LESS

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Permutation Equivalence Problem (PEP)
Given $G, G' \in \mathbb{F}_q^{k \times n}$, find $\varphi \in S_n$
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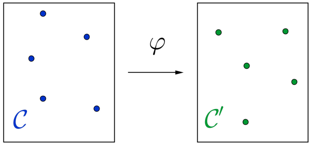

- LEP used in: LESS

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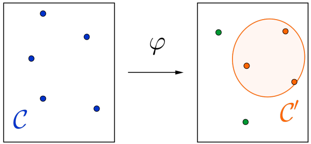
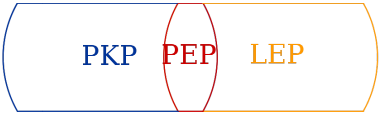
Permuted Kernel Problem (PKP)

Given $G \in \mathbb{F}_q^{k\times n}$, $G' \in \mathbb{F}_q^{k'\times n}$, find perm. matrix P s.t. $\langle G' \rangle \subset \langle GP \rangle$.





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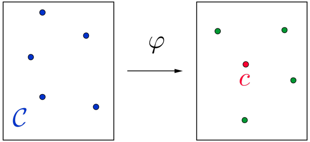
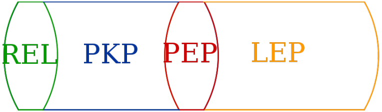
Relaxed PKP

Given $G \in \mathbb{F}_q^{k \times n}$, $G' \in \mathbb{F}_q^{k' \times n}$, find $x \in \mathbb{F}_q^k$, perm. P s.t. $xGP \in \langle G' \rangle$.





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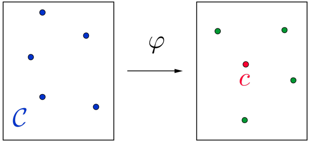

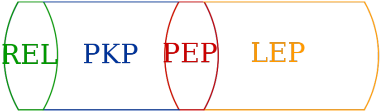

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4. Open problem: Hardness of Code Equivalence

- How hard is LEP/ relaxed PKP?
- How to solve LEP/ relaxed PKP?

Connections to Other Problems

Graph Isomorphism (GI)

Given $\mathcal{G} = (V, E), \mathcal{G}' = (V, E')$, find $f : V \to V$, s.t. $\{u, v\} \in E \leftrightarrow \{f(u), f(v)\} \in E'$.

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- Hardness?

→ Assume quasi-polynomial (Babai)

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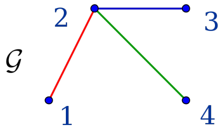

$|E| = k$

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$G = [\mathrm{Id}_k \ \mathrm{Id}_k \ \mathrm{Id}_k \ D]$$

- sub-GI NP-hard

→ PKP NP-hard

→ hardness LEP ≥ PEP ≥ GI

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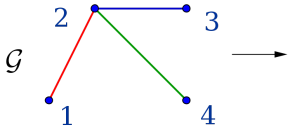

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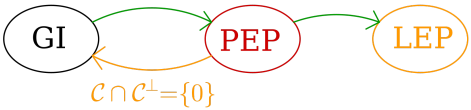

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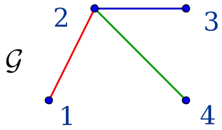

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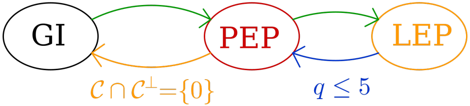

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→ if $q \leq 5$ LEP = PEP

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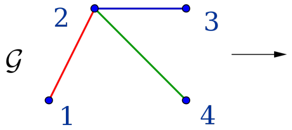

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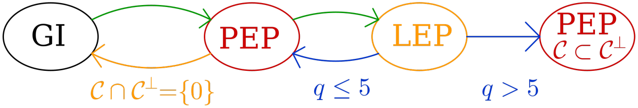

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How to solve Code Equivalence

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find permutation π $\updownarrow$

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other solvers?

Questions?

Open Problems

1. Distinguish Goppa codes
2. Find code for Hash & Sign
3. Hardness of Rank SDP
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Announcement:

CBCrypto 2024

May 25-26 in Zurich

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Slides

Thank you!