Formalising Semantics for Expected Running Time of Probabilistic Programs (Rough Diamond)

Johannes Hölzl

TU München, Germany

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- Probabilistic programs (pGCL) + expected running time.

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  - Coupon Collector

clarified semantics

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clarified semantics – different proofs

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clarified semantics – different proofs – fixed proofs

Probabilistic Guarded Command Language (pGCL)

$$\sigma \; \text{pgcl} \quad = \quad \bot$$

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Probabilistic Guarded Command Language (pGCL)

$$\sigma \; \text{pgcl} \quad = \quad \bot \mid € \mid ϟ$$

Probabilistic Guarded Command Language (pGCL)

$$\sigma \text{ pgcl} \quad = \quad \bot \mid € \mid ϟ$$
$$\mid \quad x :\sim \mathcal{D} \quad \text{or} \quad x := \text{expr} \qquad \text{"Assign } (\sigma \Rightarrow \sigma \text{ pmf)"}$$

Probabilistic Guarded Command Language (pGCL)

$$
\begin{aligned}
\sigma\ \text{pgcl} \quad &= \quad \bot \mid \text{€} \mid \text{ϟ} \\
&\mid \quad x :\sim \mathcal{D} \quad \text{or} \quad x := \text{expr} \qquad \text{"Assign } (\sigma \Rightarrow \sigma \text{ pmf)"} \\
&\mid \quad p_1 \,;\, p_2
\end{aligned}
$$

Probabilistic Guarded Command Language (pGCL)

$$
\begin{array}{rcll}
\sigma \text{ pgcl} & = & \bot \mid \text{€} \mid \text{ϟ} & \\
& \mid & x :\sim \mathcal{D} \quad \text{or} \quad x := \text{expr} & \text{"Assign } (\sigma \Rightarrow \sigma \text{ pmf)"} \\
& \mid & p_1 \; ; \; p_2 & \\
& \mid & p_1 \mid p_2 &
\end{array}
$$

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 & \mid & x :\sim \mathcal{D} \quad \text{or} \quad x := \text{expr} & \text{"Assign } (\sigma \Rightarrow \sigma \text{ pmf})\text{"} \\
 & \mid & p_1 \, ; \, p_2 & \\
 & \mid & p_1 \mid p_2 & \\
 & \mid & \text{ITE } g \; p_1 \; p_2 & g :: \sigma \Rightarrow \text{bool}
\end{array}
$$

Probabilistic Guarded Command Language (pGCL)

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\sigma\ \text{pgcl} & = & \bot \mid \text{€} \mid \text{ϟ} & \\
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 & \mid & p_1 \mid p_2 & \\
 & \mid & \text{ITE } g\ p_1\ p_2 & g :: \sigma \Rightarrow \text{bool} \\
 & \mid & \text{WHILE } g \text{ DO } p &
\end{array}
$$

Denotational Semantics (Expected Running Time)

$\mathsf{ert} :: \sigma\ \mathsf{pgcl} \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0}) \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0})$

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Values we want assigned to a *terminal state*

Denotational Semantics (Expected Running Time)

Values computed for the a *starting state*

$$\text{ert} :: \sigma\ \text{pgcl} \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0}) \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0})$$

Values we want assigned to a *terminal state*

Denotational Semantics (Expected Running Time)

$$\text{ert} :: \sigma \text{ pgcl} \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0}) \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0})$$

$$\text{ert } \bot \quad c \quad = \quad c$$

Denotational Semantics (Expected Running Time)

$$\text{ert} :: \sigma \text{ pgcl} \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0}) \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0})$$

$$\text{ert } \bot \quad c = c$$

$$\text{ert } \text{€} \quad c = 1 + c$$

Denotational Semantics (Expected Running Time)

$$\mathrm{ert} :: \sigma\ \mathrm{pgcl} \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0}) \Rightarrow (\sigma \Rightarrow \overline{\mathbb{R}}_{\geq 0})$$

$$\begin{aligned}
\mathrm{ert}\ \bot\ c &= c \\
\mathrm{ert}\ \text{€}\ c &= 1 + c \\
\mathrm{ert}\ \lightning\ c &= 0
\end{aligned}$$

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\text{ert}\ (\text{Assign}\ u)\ & c &=&\ 1 + \left(\lambda x.\ \int_y c\ y\ d(u\ x)\right)
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\text{ert}\ (\text{Assign}\ u)\ & c = 1 + \left(\lambda x.\ \int_y c\ y\ d(u\ x)\right) \\
\text{ert}\ (p_1;\ p_2)\ & c = \text{ert}\ p_1\ (\text{ert}\ p_2\ c)
\end{aligned}$$

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\text{ert}\ (p_1;\ p_2)\ && c &= \text{ert}\ p_1\ (\text{ert}\ p_2\ c) \\
\text{ert}\ (p_1 \mid p_2)\ && c &= \text{ert}\ p_1\ c \sqcup \text{ert}\ p_2\ c
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&\text{ert}\ (\text{ITE}\ g\ p_1\ p_2) && c &&= && 1 + (\lambda x.\ \text{if}\ g\ x\ \text{then}\ \text{ert}\ p_1\ c\ x\ \text{else}\ \text{ert}\ p_2\ c\ x)
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\text{ert}\ \bot & c = & c \\
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\text{ert}\ (p_1 \mid p_2) & c = & \text{ert}\ p_1\ c \sqcup \text{ert}\ p_2\ c \\
\text{ert}\ (\text{ITE}\ g\ p_1\ p_2) & c = & 1 + (\lambda x.\ \text{if}\ g\ x\ \text{then}\ \text{ert}\ p_1\ c\ x\ \text{else}\ \text{ert}\ p_2\ c\ x) \\
\text{ert}\ (\text{WHILE}\ g\ \text{DO}\ p) & c = & \text{lfp}\ (\lambda W x. 1 + \text{if}\ g\ x\ \text{then}\ \text{ert}\ p\ W x\ \text{else}\ c\ x)
\end{array}$$

Interjection: Markov decision processes

Automata with probabilistic and non-deterministic choice

$K :: \sigma \Rightarrow \sigma$ pmf set $\qquad K\,s \neq \emptyset$

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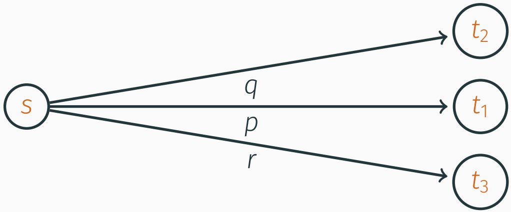

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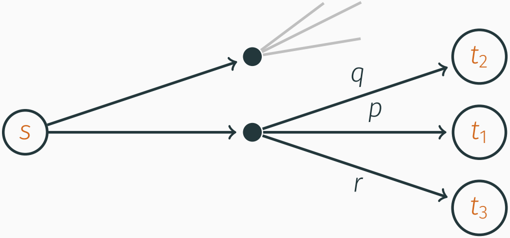

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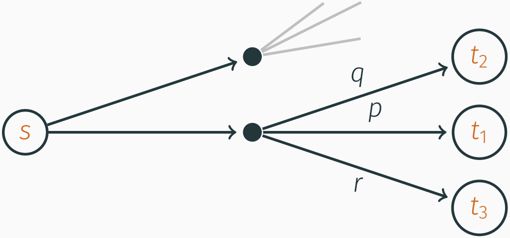

Construct maximal expectation $\hat{\mathbb{E}}_s[f]$ of a (cost) function f:

$$\hat{\mathbb{E}}_s[f] = \bigsqcup_{D \in K\, s} \int_t \hat{\mathbb{E}}_t[\lambda\omega.\ f\,(t \cdot \omega)]\ \mathrm{d}D$$

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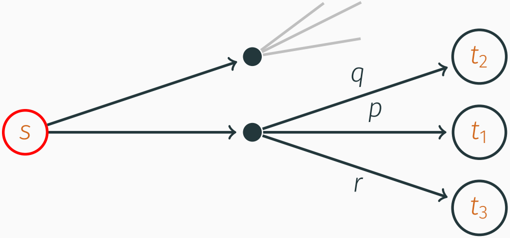

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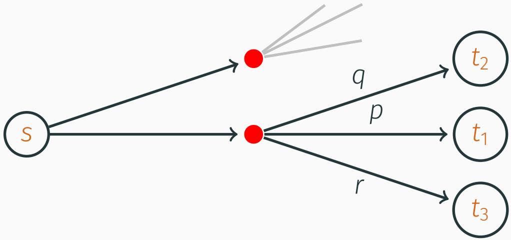

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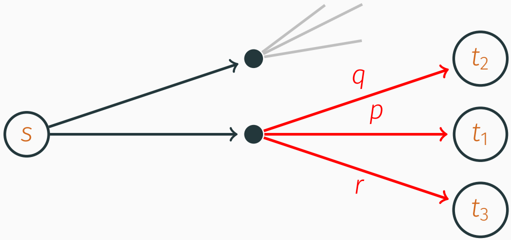

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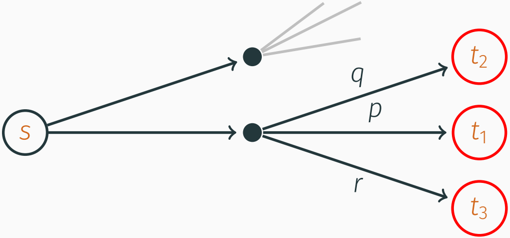

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Operational Semantics

$\mathsf{K} :: (\sigma\ \mathsf{pgcl} \times \sigma) \Rightarrow (\sigma\ \mathsf{pgcl} \times \sigma)\ \mathsf{pmf\ set}$

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$K :: (\sigma \text{ pgcl} \times \sigma) \Rightarrow (\sigma \text{ pgcl} \times \sigma) \text{ pmf set}$

$K(\bot, \quad s) = \ll \bot, s \gg$

$K(\text{€}, \quad s) = \ll \bot, s \gg$

$K(\text{ϟ}, \quad s) = \ll \text{ϟ}, s \gg$

Operational Semantics

$$\mathsf{K} :: (\sigma\ \mathsf{pgcl} \times \sigma) \Rightarrow (\sigma\ \mathsf{pgcl} \times \sigma)\ \mathsf{pmf\ set}$$

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$$\mathsf{K}(\text{↯}, \quad s) = \ll \text{↯}, s \gg$$

$$\mathsf{K}(\mathsf{Assign}\ u, \quad s) = \left[\lambda s'.\ (\bot, s')\right]\ \{u\ s\}$$

Operational Semantics

$$K :: (\sigma \text{ pgcl} \times \sigma) \Rightarrow (\sigma \text{ pgcl} \times \sigma) \text{ pmf set}$$

$$K(\bot, \qquad s) = \ll \bot, s \gg$$

$$K(€, \qquad s) = \ll \bot, s \gg$$

$$K(ϟ, \qquad s) = \ll ϟ, s \gg$$

$$K(\text{Assign } u, \qquad s) = \left[\lambda s'.\ (\bot, s')\right] \{u\ s\}$$

$$K(p_1 \mid p_2, \qquad s) = \ll p_1, s \gg \cup \ll p_2, s \gg$$

Operational Semantics

$$\mathsf{K} :: (\sigma\ \mathsf{pgcl} \times \sigma) \Rightarrow (\sigma\ \mathsf{pgcl} \times \sigma)\ \mathsf{pmf\ set}$$

$$\begin{aligned}
\mathsf{K}(\bot,\quad & s) = \ll \bot, s \gg \\
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\mathsf{K}(\mathsf{Assign}\ u,\quad & s) = \left[\lambda s'.\ (\bot, s')\right]\ \{u\ s\} \\
\mathsf{K}(p_1 \mid p_2,\quad & s) = \ll p_1, s \gg \cup \ll p_2, s \gg \\
\mathsf{K}(\mathsf{ITE}\ g\ p_1\ p_2,\quad & s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p_1, s \gg\ \mathsf{else}\ \ll p_2, s \gg \\
\mathsf{K}(\mathsf{WHILE}\ g\ \mathsf{DO}\ p,\ & s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p;\ \mathsf{WHILE}\ g\ \mathsf{DO}\ p, s \gg\ \mathsf{else}\ \ll \bot, s \gg
\end{aligned}$$

Operational Semantics

$$\mathsf{K} :: (\sigma\ \mathsf{pgcl} \times \sigma) \Rightarrow (\sigma\ \mathsf{pgcl} \times \sigma)\ \mathsf{pmf\ set}$$

$$\begin{aligned}
\mathsf{K}(\bot, \quad & s) = \ll \bot, s \gg \\
\mathsf{K}(\text{€}, \quad & s) = \ll \bot, s \gg \\
\mathsf{K}(\lightning, \quad & s) = \ll \lightning, s \gg \\
\mathsf{K}(\mathsf{Assign}\ u, \quad & s) = \left[\lambda s'.\ (\bot, s')\right] \{u\ s\} \\
\mathsf{K}(p_1 \mid p_2, \quad & s) = \ll p_1, s \gg \cup \ll p_2, s \gg \\
\mathsf{K}(\mathsf{ITE}\ g\ p_1\ p_2, \quad & s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p_1, s \gg\ \mathsf{else}\ \ll p_2, s \gg \\
\mathsf{K}(\mathsf{WHILE}\ g\ \mathsf{DO}\ p, \ & s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p;\ \mathsf{WHILE}\ g\ \mathsf{DO}\ p, s \gg\ \mathsf{else}\ \ll \bot, s \gg \\
\mathsf{K}(p_1;\ p_2, \quad & s) = \left[\lambda \begin{array}{ll} (\bot, s'). & (p_2, s') \\ (\lightning, s'). & (\lightning, s') \\ (p', s'). & (p';\ p_2, s') \end{array}\right] \mathsf{K}(p_1, s)
\end{aligned}$$

Operational Semantics

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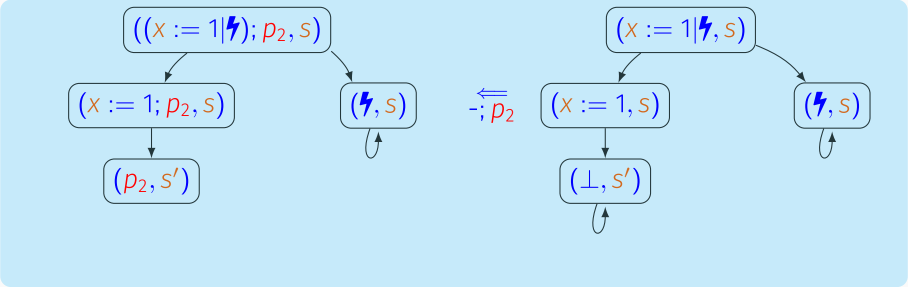

$\mathsf{K}(\mathsf{ITE}\ g\ p_1\ p_2,\ s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p_1,s \gg\ \mathsf{else}\ \ll p_2,s \gg$

$\mathsf{K}(\mathsf{WHILE}\ g\ \mathsf{DO}\ p,\ s) = \mathsf{if}\ g\ s\ \mathsf{then}\ \ll p;\ \mathsf{WHILE}\ g\ \mathsf{DO}\ p,s \gg\ \mathsf{else}\ \ll \bot,s \gg$

$$\mathsf{K}(p_1;\ p_2,\ s) = \left[\lambda \begin{array}{ll} (\bot, s'). & (p_2, s') \\ (\lightning, s'). & (\lightning, s') \\ (p', s'). & (p';\ p_2, s') \end{array}\right] \mathsf{K}(p_1, s)$$

Traces

Trace	$\mathsf{cost}_{\mathsf{stream}}$
$((x := 1 \mid \lightning); y := 0, s)$	0

Cost per stream:

$$\mathsf{cost}_{\mathsf{stream}}\ f\ (s \cdot \omega) \stackrel{\mathsf{lfp}}{=} \mathsf{cost}\ f\ s\ (\mathsf{cost}_{\mathsf{stream}}\ f\ \omega)$$

Traces

Trace	$\mathsf{cost}_{\mathsf{stream}}$
$((x := 1 \mid ⚡); y := 0, s)$	0
$\Downarrow$	$+$
$(x := 1; y := 0, s)$	1

Cost per stream:

$$\mathsf{cost}_{\mathsf{stream}}\ f\ (s \cdot \omega) \stackrel{\mathrm{lfp}}{=} \mathsf{cost}\ f\ s\ (\mathsf{cost}_{\mathsf{stream}}\ f\ \omega)$$

Traces

Trace	$\mathrm{cost}_{\mathrm{stream}}$
$((x := 1 \mid \lightning); y := 0, s)$	0
$\Downarrow$	$+$
$(x := 1; y := 0, s)$	1
$\Downarrow$	$+$
$(y := 0, s')$	1

Cost per stream:

$$\mathrm{cost}_{\mathrm{stream}}\ f\ (s \cdot \omega) \overset{\mathrm{lfp}}{=} \mathrm{cost}\ f\ s\ (\mathrm{cost}_{\mathrm{stream}}\ f\ \omega)$$

Traces

Trace	$\mathsf{cost}_{\mathsf{stream}}$
$((x := 1 \mid ⚡); y := 0, s)$	0
$\Downarrow$	$+$
$(x := 1; y := 0, s)$	1
$\Downarrow$	$+$
$(y := 0, s')$	1
$\Downarrow$	$+$
$(\bot, s'')$	$f\, s''$
$\vdots$	

Cost per stream:

$$\mathsf{cost}_{\mathsf{stream}}\; f\; (s \cdot \omega) \overset{\mathsf{lfp}}{=} \mathsf{cost}\; f\; s\; (\mathsf{cost}_{\mathsf{stream}}\; f\; \omega)$$

Interjection: Least Fixed Points

Theorem (Transfer rule for least fixed points)

$$\frac{\sqcup\text{-continuous } \alpha, f, g \qquad \alpha \perp = \perp \qquad \alpha \circ f = g \circ \alpha}{\alpha(\mathsf{lfp}\ f) = \mathsf{lfp}\ g}$$

Theorem (Transfer rule for least fixed points)

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\alpha(\mathsf{lfp}\ f) &= \alpha \circ f \circ f \circ f \circ f \circ f \circ \cdots \circ \bot \\
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&= \mathsf{lfp}\ g
\end{aligned}$$

Interjection: Least Fixed Points

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$\alpha\, f = \hat{\mathbb{E}}_s[f]$ for f Borel-measurable

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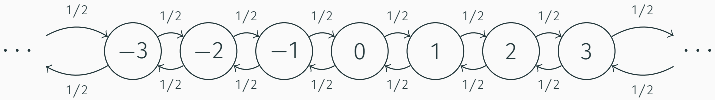
- Central to our proof: Expectation as fixed point
- [KKMO 2016]: Expectation as sums over all paths

Simple Symmetric Random Walk (ssrw)



- $\Pr_i(\Diamond j) = 1$
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  for $i \neq j$: $H\ i\ j =$

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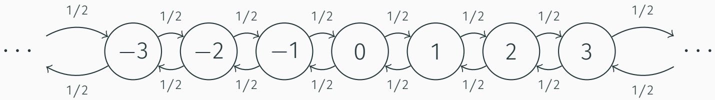

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Simple Symmetric Random Walk – [KKMO]

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How do **Kaminski, Katoen, Matheja, and Olmedo** prove it?

- $\mathsf{ert}\ (\mathsf{ssrw}\ j) \perp i$ has lower ω-invariant $I_{n+1} \leq \mathsf{ert}\ \mathsf{STEP}\ I_n$:

$$I_n = 1 + [\![0 < x \leq n]\!] \cdot \infty$$

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Fails for $x = 1$ and $n = 0$.

Simple Symmetric Random Walk – our Solution

Operational semantics – trace representation

Use $H\ i\ j = \hat{\mathbb{E}}_{(\text{ssrw}\ j, i)}[\text{cost}_{\text{stream}}]$ to prove

$$H\ i\ j = H\ i\ k + H\ k\ j \quad \text{for} \quad i \leq k \leq j$$

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Coupon Collector

$$
\begin{aligned}
&x := 0, i := 0, cp := [\overbrace{F, \ldots, F}^{N \text{ times}}];\\
&\text{WHILE } x < N \text{ DO}\\
&\quad \text{WHILE } cp[i] \text{ DO}\\
&\quad\quad i :\sim \mathcal{U}(\{1, \ldots, N\});\\
&\quad cp[i] := T, x := x + 1
\end{aligned}
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$$\text{ert } \mathsf{CC}_N \ 0 \ s = 2 + N \cdot \left(4 + 2 \sum_{i=1}^{N} \frac{1}{i}\right).$$

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\quad \text{WHILE } cp[i] \text{ DO} & cp[i] \to b & \quad \text{WHILE } b \text{ DO} \\
\quad\quad i :\sim \mathcal{U}(\{1, \ldots, N\}); & |cp| = x & \quad\quad b :\sim \mathcal{B}(x/N); \\
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$cp[i] \to b$
$|cp| = x$

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KKMO use loop invariants to prove running time

Related Work

Different pGCL formalizations already exist:

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Related Work

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Cock [SSV 2012]	VCG in Isabelle/HOL, shallow

Conclusion & Future Work

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- Explore: Probabilistic relational Hoare logic (pRHL) + non-determinism