# Proving Inequalities over Reals With Computation in Isabelle/HOL

Johannes Hölzl

Technische Universität München
Institut für Informatik
Theorem Proving Group

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# Goal

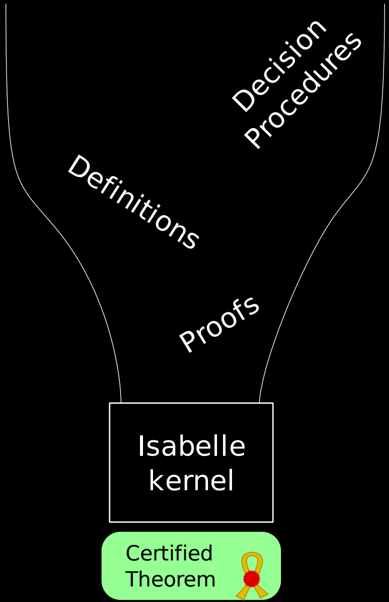
**Problem** Prove inequalities of reals

$$|\pi - 3.14159265358979323 85| < 10^{-18}$$

$$0 \le x \land x \le 1 \longrightarrow \arctan x < \frac{8}{10}$$

**Solution** Compute formula using interval arithmetic

# Isabelle




- Kernel implements only primitives
  - Non-recursive constant definitions
  - Variable instantiation
  - …
- Everything else must use the kernel:
  - Proof methods
  - Definitions of: Inductive predicates, Functions, Datatypes
  - Decision procedures
  - …

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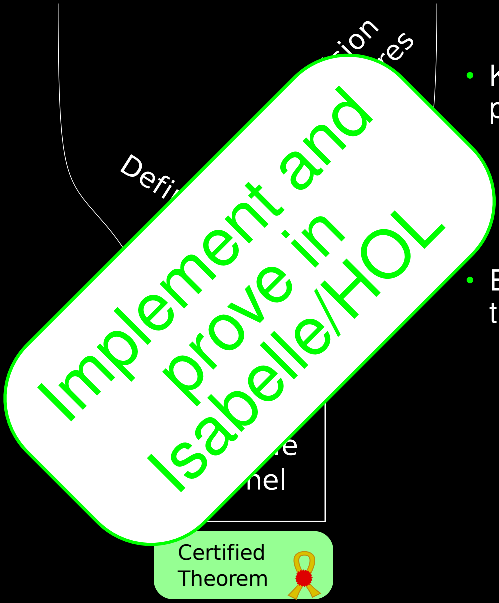


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  - …

# Related Work

- Marc Daumas, David R. Lester, and César Muñoz
  Verified Real Number Calculations: A Library for Interval Arithmetic
  *IEEE Transactions on Computers*, 58(2):226–237, 2009.
- Guillaume Melquiond
  Proving bounds on real-valued functions with computations
  *Proceedings of the 4th International Joint Conference on Automated Reasoning*, volume 5195 of *Lectures Notes in Artificial Intelligence*, pages 2–17.

## Proof

$$0 \le x \wedge x \le 1 \longrightarrow \mathit{arctan}\ x < 8/10$$

$\Uparrow$ Reification

$$\mathit{interp}\ (\mathit{Less}\ (\mathit{Arctan}\ \ldots)\ (\mathit{Mult}\ \ldots\ \ldots))\ [x]$$

$\Uparrow$ Approximate

$$\mathit{approx}\ 10\ (\mathit{Less}\ (\mathit{Arctan}\ \ldots)\ (\mathit{Mult}\ \ldots\ \ldots))\ [\lfloor(0,\ 1)\rfloor]\ []$$

$\Uparrow$ Evaluate

$$\mathit{True}$$

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# Reification

We want to define:

$$
\begin{aligned}
\mathit{approx}\,(\mathit{sin}\ x) &= \ldots \\
\mathit{approx}\,(\mathit{cos}\ x) &= \ldots \\
\mathit{approx}\,(x + y) &= \ldots \\
\mathit{approx}\,(x \cdot y) &= \ldots
\end{aligned}
$$

- Terms over reals not a datatype
- Cannot operate on every term

  For example: $\sum_{n=0}^{\infty} x^n$

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- Terms over reals not a datatype
- Cannot operate on every term

  For example: $\sum_{n=0}^{\infty} x^n$

**Solution:** Introduce a datatype representing *real* terms

# Reification

- *floatarith* represents *real* terms

  $$floatarith = Add\ floatarith\ floatarith \mid Sin\ floatarith \mid \ldots$$

- *interp* interprets *floatarith* as *real*:

  $$\begin{aligned} interp\ (Arctan\ a)\ vs &= arctan\ (interp\ a\ vs) \\ interp\ (Add\ a\ b)\ vs &= interp\ a\ vs + interp\ b\ vs \\ &\vdots \end{aligned}$$

- `reify`, generates *floatarith* term from *real* term

  $$interp\ (Arctan\ (Var\ (0 :: 'a)))\ [x] = arctan\ x$$

$$
\begin{aligned}
&\mathit{interp} && (\mathit{Less}\ a\ b) && \mathit{vs} && = && \mathit{interp}\ a\ \mathit{vs} < \mathit{interp}\ b\ \mathit{vs}\\
&\mathit{interp} && (\mathit{Mult}\ a\ b) && \mathit{vs} && = && \mathit{interp}\ a\ \mathit{vs} \cdot \mathit{interp}\ b\ \mathit{vs}\\
&\mathit{interp} && (\mathit{Arctan}\ a) && \mathit{vs} && = && \mathit{arctan}\ (\mathit{interp}\ a\ \mathit{vs})\\
&\mathit{interp} && (\mathit{Inverse}\ a) && \mathit{vs} && = && \mathit{inverse}\ (\mathit{interp}\ a\ \mathit{vs})\\
&\mathit{interp} && (\mathit{Num}\ f) && \mathit{vs} && = && f\\
&\mathit{interp} && (\mathit{Var}\ n) && \mathit{vs} && = && \mathit{vs}\ !\ n\\
&\mathit{interp} && (\mathit{Mult}\ a\ (\mathit{Inverse}\ b)) && \mathit{vs} && = && \frac{\mathit{interp}\ a\ \mathit{vs}}{\mathit{interp}\ b\ \mathit{vs}}
\end{aligned}
$$

- Syntax
- Semantics

$$\begin{array}{l} interp\ a\ vs = t_1 \ \wedge \\ interp\ b\ vs = t_2 \end{array} \longrightarrow \quad interp\ (Less\ a\ b)\ vs \quad = \quad (t_1 < t_2)$$

$$\begin{array}{llll}
interp & (Mult\ a\ b) & vs & = & interp\ a\ vs \cdot interp\ b\ vs \\
interp & (Arctan\ a) & vs & = & arctan\ (interp\ a\ vs) \\
interp & (Inverse\ a) & vs & = & inverse\ (interp\ a\ vs) \\
interp & (Num\ f) & vs & = & f \\
interp & (Var\ n) & vs & = & vs\ !\ n \\
interp & (Mult\ a\ (Inverse\ b)) & vs & = & \frac{interp\ a\ vs}{interp\ b\ vs}
\end{array}$$

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$$interp\ a\ vs = t_1 \land interp\ b\ vs = t_2 \longrightarrow interp\ (Less\ a\ b)\ vs = (t_1 < t_2)$$

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$$\begin{aligned}
interp\ (Arctan\ a)\ vs &= arctan\ (interp\ a\ vs) \\
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interp\ (Num\ f)\ vs &= f \\
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interp\ (Mult\ a\ (Inverse\ b))\ vs &= \frac{interp\ a\ vs}{interp\ b\ vs}
\end{aligned}$$

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$$\frac{interp\ a\ vs = t_1\ \wedge\ interp\ b\ vs = t_2}{} \longrightarrow interp\ (Less\ a\ b)\ vs = (t_1 < t_2)$$

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$$interp\ a\ vs = t \longrightarrow interp\ (Arctan\ a)\ vs = arctan\ t$$

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**Goal:**

$$interp\ \ (Less\ (Arctan(Var\ 0))\ \ (Mult(Num\ 8)(Inverse(Num\ 10)))\ )\ \ [x]$$

$$\Leftrightarrow$$

$$arctan\ x\ <\ \frac{8}{10}$$

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$$\mathit{interp}\ (\mathit{Less}\ (\mathit{Arctan}\ \ldots)\ (\mathit{Mult}\ \ldots\ \ldots))\ [x]$$

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$$\mathit{True}$$

# Approximate

- Done by *approx*
- Interval arithmetic
- Interval boundaries: *float*s
- Resulting intervals must not overlap (provide precision parameter)

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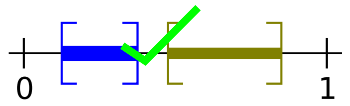


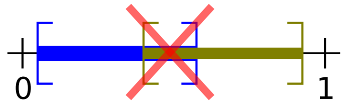

# Interval arithmetic

- $x$ represented by $l$ and $u$:
  $x \in \{\ l \,..\, u\ \}$
- Function $f$ implemented as *ub-f*, *lb-f*:
  $\forall\ x \in \{\ l \,..\, u\ \}.\ f\ x \in \{\ lb\text{-}f\,(l,u) \,..\ ub\text{-}f\,(l,u)\ \}$
- Basic arithmetic operators easy to implement:

$$
\begin{aligned}
(x_l, x_u) + (y_l, y_u) &= (x_l + y_l, x_u + y_u) \\
-(x_l, x_u) &= (-x_u, -x_l) \\
(x_l, x_u) \cdot (y_l, y_u) &= (\mathit{min}\,\{x_l \cdot y_l, x_u \cdot y_l, x_l \cdot y_u, x_u \cdot y_u\}, \\
&\quad\ \ \mathit{max}\,\{x_l \cdot y_l, x_u \cdot y_l, x_l \cdot y_u, x_u \cdot y_u\}) \\
(x_l, x_u)^{-1} &= ({x_u}^{-1}, {x_l}^{-1}) \\
&\quad \text{when } 0 \notin \{x_l \,..\, x_u\}
\end{aligned}
$$

# Floating-point numbers

- Represent interval boundaries
- Definition:

$$\textbf{datatype}\ \mathit{float} = \mathit{Float}\ \mathit{int}\ \mathit{int}$$

$$\mathit{real}\,(\mathit{Float}\ m\ e) = m \cdot 2^e$$

- Equations for addition, multiplication and minus:

$$\begin{aligned} \mathit{real}\,(a+b) &= \mathit{real}\ a + \mathit{real}\ b \\ \mathit{real}\,(a \cdot b) &= \mathit{real}\ a \cdot \mathit{real}\ b \\ \mathit{real}\,(-a) &= -\mathit{real}\ a \end{aligned}$$

- Bounding functions for division:

$$\begin{aligned} \mathit{real}\,(\mathit{lb\text{-}div}\ \mathit{prec}\ a\ b) &\le \frac{\mathit{real}\ a}{\mathit{real}\ b} \\ \mathit{real}\,(\mathit{ub\text{-}div}\ \mathit{prec}\ a\ b) &\ge \frac{\mathit{real}\ a}{\mathit{real}\ b} \end{aligned}$$

(Why? $\frac{1}{3} = 0.3333\ldots$ !)

# Implementation of transcendental functions

($\pi$, arctan, sin, cos, exp, and ln)

- Use Taylor-series
- *But:* Not applicable in the entire domain
  $\Longrightarrow$ Apply some transformations

# Example of a transcendental function: arctan

**Definition:** $arctan\ y = (THE\ x \in \{-\frac{\pi}{2} <..< \frac{\pi}{2}\}.\ tan\ x = y)$

**Lemma:** $|x| \leq 1 \Longrightarrow arctan\ x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k \cdot 2 + 1} \cdot x^{k \cdot 2 + 1}$

Apply transformations:

$$\arctan(x) = \begin{cases} -\arctan(-x) & \text{if } x < 0 \\ \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k \cdot 2 + 1} \cdot x^{k \cdot 2 + 1} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 \cdot \arctan\left(\frac{x}{1 + \sqrt{1 + x^2}}\right) & \text{if } \frac{1}{2} < x \leq 2 \\ \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right) & \text{else} \end{cases}$$

# Compute arctan

$$\arctan(x) = \begin{cases} -\arctan(-x) & \text{if } x < 0 \\ \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k \cdot 2 + 1} \cdot x^{k \cdot 2 + 1} & \text{if } 0 \le x \le \frac{1}{2} \\ 2 \cdot \arctan\left(\frac{x}{1 + \sqrt{1 + x^2}}\right) & \text{if } \frac{1}{2} < x \le 2 \\ \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right) & \text{else} \end{cases}$$

ub-arctan prec x =
(

if x < 0 then – lb-arctan prec (– x)

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ub-arctan prec x =
(let ub = $\lambda$x. x $\cdot$ ub-arctan-horner prec (get-odd (prec div 4 + 1)) 1 ($x^2$);

in if x < 0 then − lb-arctan prec (− x)
else if x ≤ $\frac{1}{2}$ then ub x

)

# Compute arctan

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else if x ≤ 2
then let x' = ub-div prec x (1 + the (lb-sqrt prec (1 + $x^2$)))
in if 1 < x' then ub-pi prec $\cdot$ $\frac{1}{2}$ else 2 $\cdot$ ub x'
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ub-arctan prec x =
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lb = $\lambda$x. x $\cdot$ lb-arctan-horner prec (get-even (prec div 4 + 1)) 1 ($x^2$)
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in if 1 < x' then ub-pi prec $\cdot \frac{1}{2}$ else 2 $\cdot$ ub x'
else ub-pi prec $\cdot \frac{1}{2}$ − lb (lb-div prec 1 x))

# Correctness

$\forall x \in \{\text{real lx} \,..\, \text{real ux}\}.$
$\quad \text{arctan } x \in \{\text{real (lb-arctan prec lx)} \,..\, \text{real (ub-arctan prec ux)}\}$

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When proved for each function:

$$n = \mathit{length}\ \mathit{xs} \wedge \mathit{approx}\ \mathit{prec}\ f\ (\mathit{replicate}\ n\ \mathit{None})\ \mathit{ss} \longrightarrow \mathit{interp}\ f\ \mathit{xs}$$

## Proof

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## Proof

$$0 \le x \wedge x \le 1 \longrightarrow \mathit{arctan}\ x < 8/10$$

$\Uparrow$ Reification

$$\mathit{interp}\ (\mathit{Less}\ (\mathit{Arctan}\ \ldots)\ (\mathit{Mult}\ \ldots\ \ldots))\ [x]$$

$\Uparrow$ Approximate + Extensions

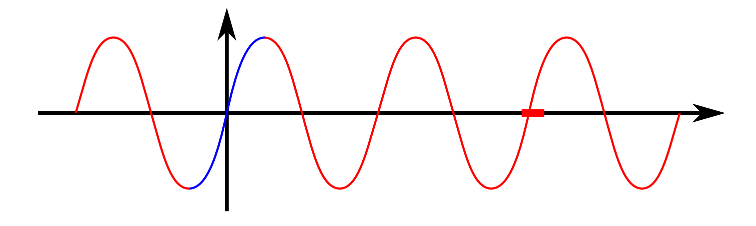
$$\mathit{approx}\ 10\ (\mathit{Less}\ (\mathit{Arctan}\ \ldots)\ (\mathit{Mult}\ \ldots\ \ldots))\ [\lfloor(0,\ 1)\rfloor]\ []$$

$\Uparrow$ Evaluate

*True*

# Argument Reduction

**Problem:** sin only monotone in $\{-\frac{\pi}{2}..\frac{\pi}{2}\}$



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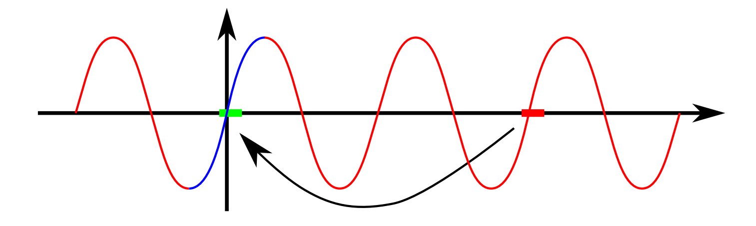
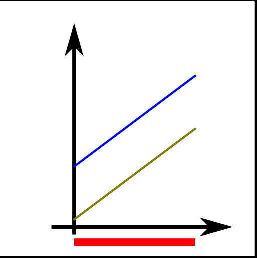


**Solution:** Shift input interval by $k \cdot \pi$.
(Same for cos)

# Interval Splitting



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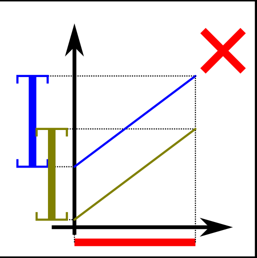

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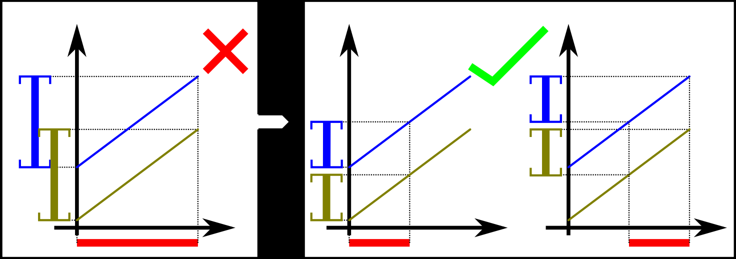

# Taylor Series Expansion

**Problem:** Dependency effect
$\forall x \in X = \{0..1\}$:

$$x - x \leq 0$$

$$X - X = \{-1..1\} \neq \{0..0\}$$

# Taylor Series Expansion

**Problem:** Dependency effect

$\forall x \in X = \{0..1\}$:

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$$X - X = \{-1..1\} \neq \{0..0\}$$

**Solution:**

- Avoid $X - X$
- $f(X) = X - X$
- Taylor series expansion: $f(X) \Longrightarrow f(c) + f'(X) \cdot (X - c)$
- $X - X \Longrightarrow (c - c) + (0 - 0) \cdot (X - c)$

# Orbiting satellite

**lemma** orbital-period:
**assumes**
$G = 6.67428 / 10\hat{~}11$ — *Gravitational constant*
$r = 35786500$ — *Altitude of satellite*
$v = 3593.71$ — *Speed of satellite*
$M = 6 \cdot 10\hat{~}24$ — *Mass of earth*

$\mu = G \cdot M$
$p = (r \cdot v)\hat{~}2 / \mu$
$e = (|v|\hat{~}2 / \mu) \cdot r - 1$
**shows**
$| 2 \cdot pi \cdot sqrt((p / (1 - e\hat{~}2))\hat{~}3 / \mu) - 24 \cdot 60 \cdot 60 | < 1$

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$\mu$ = G · M
p = (r · v)ˆ2 / $\mu$
e = (|v|ˆ2 / $\mu$) · r – 1
**shows**
| 2 · pi · sqrt((p / (1 – eˆ2))ˆ3 / $\mu$) – 24 · 60 · 60 | < 1

**by** (approximation 60) □

Finished!